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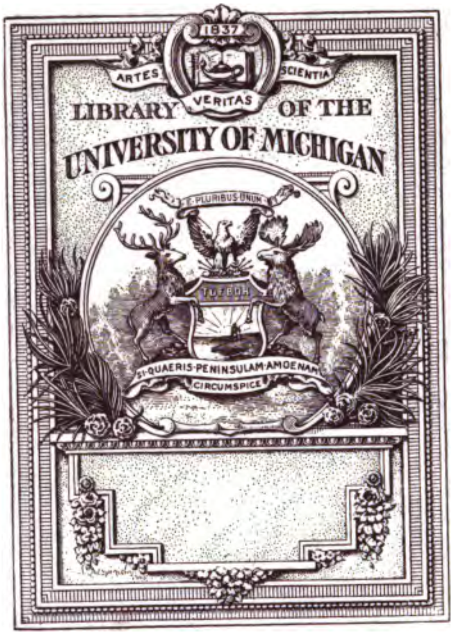
PROBLEMS AND STUDIES

PART 2

OF

THE ACCOUNTANCY OF INVESTMENT

SPRAGUE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Studies in Business

First Series No. 4

PROBLEMS AND STUDIES
IN
THE ACCOUNTANCY OF INVESTMENT

BY CHARLES EZRA SPRAGUE, A.M., PH.D., C.P.A.

AUTHOR OF
"THE ACCOUNTANCY OF INVESTMENT," (TO WHICH THIS IS A SUPPLEMENT) AND
"EXTENDED BOND TABLES"

MDCCCXXXI

PUBLISHT FOR THE NEW YORK UNIVERSITY SCHOOL OF COMMERCE,
ACCOUNTS AND FINANCE, BY THE AUTHOR
NEW YORK, 1906

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PREFACE.

The success attained by my book entitled "The Accountancy of Investment," has induced me to prepare a companion volume, which will, in several ways, complement and supplement that work for various classes of users. For those who are systematically studying the subject without an instructor, I have presented a supply of practical problems with their solutions, enabling each to test his progress and remove his doubts, while ascertaining whether he is competent to pass an examination. His failure will send him back to the text-book to take a fresh grasp upon principles. Again, many operations can be facilitated by the skillful use of special methods, the result of experience, which will be useful in other lines of work, but which would, if inserted in the body of the work, have broken the continuity of the argument. For like reasons, some branches of the subject, such as serial and optional bonds, while sufficiently treated for the learner to understand their principles, would need to be more fully elaborated and exemplified to meet the wishes of those who are transforming their systems of accountancy to meet modern requirements and find their greatest difficulties in these very branches. The use of tables, too, needs to be taught and the means of extending them when it is requisit, and a selection of useful tables will form a companion volume.

I have been commended for clearness in my former treatise; I have tried to deserve the same praise here by simply using plain words in straightforward order.

Charles E. Sprague.

October, 1905.

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PROBLEMS AND STUDIES.

SECTION 1. SIMPLE INTEREST

"The Accountancy of Investment" Articles 11, 12

(1) What is the time in months and days from January 10th to June 12th? (2) To July 4th? (3) To September 1st? (4) What date is two months after June 30th? (5) Four months after May 31st? (6) Two months after December 31st? (7) Five months seven days after September 26th?

Article 13

(8) On a security for $54,750, interest payable semi-annually at 4% per annum, interest was last paid to November 1st. Compute the interest accrued on February 25th: (a) in the customary manner which was legal before 1892; (b) correct the odd days, making them 365ths of a year. (9) Compute the same at 4½%. (10) 5%. (11) 6%.

The One Per Cent Method

Observe that when days are reckoned as 360ths of a year, it is very useful to know how many days correspond to one per cent. If the rate is 3%, it takes 120 interest days to earn 1% interest.

At 3% the number of days for 1% is 120
At 4% " " " 90
At 4½% " " " 80
At 5% " " " 72
At 6% " " " 60
At 8% " " " 45
At 9% " " " 40

We may set down the number of days corresponding to the given rate and in line with it the principal, cutting off two places from the latter to obtain 1%. Thus, in example (8)

90 547 | 50

meaning that the interest for 90 days is $547.50. Knowing the interest for 90 days, we can build up that for 114 days (meaning 3 months 24 days). Dividing 90 by 6, gives 15; dividing 90 by 10, gives 9. Performing the same divisions on the interest as on the number of days and adding, gives the result.

90	547	50
15	91	25
9	54	75
114	693	50

Or for 5%

72	547	50
18	136	875
24	182	50
114	866	375

Rates, like 7% or 3½%, which are not exact divisors of 360 must be obtained from the exact rates by division and addition. Thus, 7% is derived by adding 1/6 to 6%, which is obtained as above.

	60	547	50
	30	273	75
	20	182	50
	4	36	50
	114	6)1040	25
		173	375
7%		1213	625

SOLUTIONS.—(1) 5mo. 2d. (2) 5mo. 24d. (3) 7mo. 22d. (Remember, the odd days are actual days.) (4) Aug. 30. (5) Sept. 30. (6) Feb. 28 or 29. (7) March 5 or 4.

(8) a, \$693.50; b, \$691.50. (9) a, \$780.19; b, \$777.94. (10) a, \$866.875; b, \$864.375. (11) a, \$1040.25; b, \$1037.25. The principal was so selected as to be divisible by 73 or 365.

SECTION 2. COMPOUND INTEREST

Articles 14, 22

(12) Find the amount of \$1 at 2% per period for 1, 2, 3, 4 and 5 periods; (13) find the present worths of \$1 at 2% for 1, 2, 3, 4 and 5 periods, rounded at 6 places. In Article 19, three methods are mentioned for finding the present worth. It will be easiest to use the third method—as the materials for it have been obtained in question (12)—and first find the present worth for 5 periods; thence for 4, 3, 2 and 1.

(14) Find the amount of \$1 at 1¾% (.0175) per period for 1, 2, 3, 4, 5 and 6 periods. (15) Find the present worth of \$1 at 1.75% per period for 6, 5, 4, 3, 2 and 1 period. (16) Find the amount and the present worth of \$1,000 for 8 periods at 1½% per period.

Articles 23, 24

(17) What is the rate of discount corresponding to 2% interest? (18) What is the rate of interest corresponding to the discount-rate .0384615+? (19) What is the compound interest of $1 for 5 periods at 2%? (20) What is the compound discount of $1 for 4 periods at 2%?

SOLUTIONS.—(12) For 5 periods, $1.104081
(13) For 5 periods, .905731
(14) For 6 periods, 1.109702
(15) For 6 periods, .901143
(16) For 8 periods, amount $1,126.493; present worth, $887.711.

(17) .0196078. Observe that these figures when divided by 2 (the rate of interest) give 1/100 of the present worth. This will test all similar computations. (18) 4%. (19) .104081. (20) .0761546.

Test of Amount and Present Worth

The amount and the present worth of the same sum for the same time and rate should, when multiplied together, give the product 1.

Problems 12 and 13 give the amount of $1 for 5 periods at 2% as 1.104081, and its present value for the same time and rate as .905731. These multiplied together should give as a product, unity. Such multiplications of decimal numbers are best performed by beginning at the *left* of the multiplier.

1.10408	1
.905731	
993672	9
5520	405
772	8567
33	12243
1	104081
1.000000	388211

The vertical line is drawn to cut off the figures beyond the 6th decimal, which have no utility except to furnish a carrying for the 6th figure. They may be dispensed with by using contracted multiplication.

Contracted Multiplication

In this process we shorten the multiplicand at each step by one figure, taking into account, however, the carrying from the rejected figures.

	1.104081
	.905731
(6 figures × 9)........	993673
(5 figures × 0)........	0
(4 figures × 5)........	5520
(3 figures × 7)........	773
(2 figures × 3)........	33
(1 figure × 1).........	1
	1.000000

We commence to "× 9" at the figure 8; the product would be 72, but we know that the rejected 1, × 9, would make it nearer 73; we therefore write 993673. In each partial product we round the last retained figure up or off by mental allowance for the carrying of the next rejected figure. The last figure will, even then, not always be exact, but may vary one or two units. In all multiplications by rounded decimals, there is a final error; this can only be allowed for by carrying the process a few places further before multiplying.

It sometimes happens in contracted multiplication that you "lose your place" and forget at what figure of the multiplicand you should begin next. This may be overcome by ticking off each figure as you have done with it; or, if you do not do this, tell off the multiplicand figures from right to left, repeating the multiplier figures from left to right.

All the foregoing results may be tested in this way.

SECTION 3. THE USE OF LOGARITHMS

Articles 28, 43

For exercise in logarithmic computations, work out again examples (12) to (17) of the previous chapter, using logarithms to the limit of the tables you may have. The logarithm of the ratio of increase, with which the operation always begins, will be found in Appendix I, Acc. of Inv.

The following examples, which are for too many periods to be worked arithmetically, may also be worked by logarithms.

Amounts of $1	(21)	1¼%, 30 periods	(26)	Present worths of $1
	(22)	1.70%, 50 periods	(27)	
	(23)	2%, 10 periods	(28)	
	(24)	2.40%, 68 periods	(29)	
	(25)	2½%, 70 periods	(30)	

SOLUTIONS.—(21) 1.45161336. (22) 2.3229916. (23) 1.21899442. (24) 5.0164565. (25) 5.63210286. (26) .68888867. (27) .4304794. (28) .82034830. (29) .19934390. (30) .17755358.

SECTION 4. ANNUITIES

Articles 44, 46

Find the amount and present worth of the following annuities:

Amount of an annuity of $1	(31)	30 periods at	1¼%	(36)	Present worth of an annuity of $1
	(32)	50 "	1.70%	(37)	
	(33)	10 "	2%	(38)	
	(34)	68 "	2.40%	(39)	
	(35)	70 "	2½%	(40)	

In any or each of the above problems (36) to (40) construct a schedule showing the gradual repayment of the $1 for a few periods, or for the entire time.

SOLUTIONS.—(31) 36.1290688. (32) 77.82304. (33) 10.949721. (34) 167.352355. (35) 185.2841144. (36) 24.888906. (37) 33.50121. (38) 8.982585. (39) 33.360671. (40) 32.897857.

SECTION 5. BOND TABLES AS ANNUITY TABLES

The author's book of "Extended Bond Tables" can be used as an Annuity Table in case of need, when a copy of such tables is not at hand, or when the figures are not sufficiently extended or the rates not close enough. It must be remembered that its results are all based on semi-annual payment of interest; the period is a half year.

Half Yearly Annuities

In the foregoing examples on Annuities where periods and rates per period are used, we shall in practice have to transform them to years and rates per annum, payable semi-annually, the rent being half as much.

30 periods, 1¼% per period, becomes 15 years at 2.50%
50 periods, 1.7% per period, becomes 25 years at 3.40%
10 periods, 2% per period, becomes 5 years at 4%
68 periods, 2.4% per period, becomes 34 years at 4.80%
70 periods, 2½% per period, becomes 35 years at 5%

As explained in Chapter VIII, the premium or discount on a bond is nothing more nor less than the present worth of an annuity.

Present Worth

Taking the second case, 25 years at 3.40%, we turn to the 5% bond table, page 88, at the line 3.40 and write down

	1,268,009.70
From the 4% table, page 54, we have	1,100,503.64
The difference	167,506.06

is the present worth of an annuity of $10,000 per annum payable semi-annually at 3.40%, for 25 years; each semi-annual payment being $5,000, or, expressed in periods, it is the present worth of an annuity of $5,000 per period, for 50 periods, at 1.7% per period.

Instead of these two rates, 4% and 5%, we might have selected 3 and 4, 3½ and 4½, 5 and 6, or any two differing by 1%.

For example:	4% bond	1,100,503.64
	3% bond	932,997.57
	Annuity worth	167,506.07

The discrepancy of one cent is owing to the rounded decimals. Either is *within* one cent.

The reason of the process may be explained as follows: On a 5% bond, net income 3.40%, the cash interest on $1,000,000 is

	50,000	
The net income is	34,000	
The difference of interest is		16,000
With a 4% bond we have		
Cash interest	40,000	
Net income	34,000	
		6,000

Hence the tables give the values of two annuities, $16,000 and $6,000, and their difference must always be the present worth of an annuity of $10,000.

Rule

RULE: The present worth of an annuity of $10,000, payable semi-annually, at a certain rate of interest earned, is equal to the difference between the values of a 4% and a 5% bond for $1,000,000 at the same rate.

If it should happen that the rent of the desired annuity were $5,000 instead of $10,000, this might be obtained at once from the values of 3 and 3½, or 3½ and 4 per cent. bonds. Similarly the difference between 3½ and 5 would value an annuity of $15,000; 3 and 5, $20,000; 3½ and 6, $25,000; 3 and 6, $30,000; 3½ and 7, $35,000; and 3 and 7, $40,000. These results would be a trifle more accurate in the last figure than those obtained by multiplying $10,000, for the multiplication of figures which have been rounded increases the error.

The value of an annuity may also be obtained by division from a single bond value, instead of taking the difference of two. We saw that the premium on a 4% bond to net 3.40% is the present worth of an annuity of $6,000; therefore, if the premium be divided by 6, it will give the value of an annuity of $1,000.

100,503.64 ÷ 6 = 16,750.61

From the value of an annuity obtained as above, the compound discount and the present value of a single sum for the same time and rate can also be obtained.

Compound Discount

Multiplying the value of the annuity by the number of units in the rate per cent. gives the compound discount of a single sum of $1,000,000.

167,506.06 × 3.4 = 569,520.60
Subtract from 1,000,000.00

and we have 430,479.40

as the present worth of $1,000,000 payable in a single sum in 15 years at 3.40% compounded semi-annually. The last figure is unreliable. In fact, the cents should be .38.

Amount

If necessary, in the absence of compound interest tables or logarithms, the *amount* of a single sum at compound interest may be obtained. The present value of $1 is .4304794; divide $1 by this, using contracted multiplication.

```
4304794)1.0000000(2322991.7
         8609588
         -------
         1390412
         1291438||  (This sign indicates contraction
         -------          or rounding)
           98974
           86096||
           -----
           12878
            8610||
           -----
            4268
            3874||  (For contracted multiplication
            ----     see Section 2, page 10)
             394
             387||
             ---
               7
               4||
               -
               3
```

Amount of Annuity

From the compound interest $1.3229917 divided by .034 (see article 51) may be obtained the amount of an annuity. Thus, all problems in compound interest are soluble through the bond-tables. (See Summary of Interest Processes, page 63.)

SECTION 6. RENT OF ANNUITY AND SINKING FUND

Rent of Annuity

Articles 65, 68

(41) What is the rent of an annuity of 30 periods valued at $1,000 if the rate of interest is 1¼%? Or, what is each term of an annuity such that its present worth is $1,000, the interest being at 1¼% and the number of periods 30? Begin with $1.

(42) Assume 50 periods and 1.70%
(43) " 10 " 2%
(44) " 68 " 2.40%
(45) " 70 " 2½%

Sinking Fund

Articles 69, 71

(46) What is the sinking fund to be reserved each period and invested at 1¼% to amount to $1,000 at the end of 30 periods?

(47), (48), (49) and (50) What sinking funds for the same data as in (42), (43), (44) and (45)?

(51) What amount should be laid aside each half year to amount to $100,000 at the end of 50 years at 4% per annum?

(52) What amount at 3%?

SOLUTIONS: (41) 40.17854. (42) 29.84967. (43) 111.32653. (44) 29.975416. (45) 30.39712. (46) 27.67854. (47) 12.84967. (48) 91.32653. (49) 5.975416. (50) 5.39712. (51) 320.27. (52) 437.06.

SECTION 7. NOMINAL AND EFFECTIV RATES

Articles 72, 75

(58) If the interest paid is 12% per annum, but paid in monthly installments, what is the effectiv annual rate? (54) To what rate, paid semi-annually, is it equal? First translate the terms into periods, and rates per period. (55) What is the rate paid semi-annually equivalent to .0099505 per quarter?

(56) If the rate is 4% per annum payable semi-annually what rate quarterly will produce the same income? (57) What rate monthly?

(58) Interest being worth 6% per annum, which is the more valuable—an income of $4,080, payable annually, or an income of $4,000, payable quarterly? and by how much?

(59) Interest being worth 5% per annum converted quarterly, what rate should be paid annually as an equivalent?

SOLUTIONS: (53) 12.68%. (54) 12.30%. (55) 4%. (56) 3.98%. (57) 3.97%. (58) The latter by $10.90. (59) 5.095%.

Monthly Compounding

Article 72

It may be interesting to know how to ascertain that 6.184% is larger than any possible rate to which 6% could be increased by compounding every second, or every millionth of a second or constantly. This limit can easily be worked by the following rule, which, however, cannot be demonstrated here:

Finding Effectiv Rate

> RULE.—Multiply the constant quantity .4342944819+, or so much as is necessary, by the nominal rate per annum expressed decimally; find in the table of logarithms this product; from the number opposite subtract 1 and the remainder is the effectiv rate.

EXAMPLE: 6% compounded continuously. .4342944819 × .06 = .026057668914.

But .02605767 is the logarithm of 1.061837; 1.061837 — 1 = .061837, which is the limit required.

The opposit rule is for finding a nominal rate which, if compounded continually, will amount to a given effectiv rate at the end of the year.

Finding Nominal Rate

RULE.—Multiply the *logarithm* of the effectiv ratio by the constant quantity 2.302585092994+, or so much of it as necessary, and the result will be the nominal rate itself.

EXAMPLE: What rate compounded continuously will amount of 6% effectiv? Log 1.06 = .02530587; .02530587 × 2.302585 = .058270.

Approximate Rules

An approximation to the rate may also be obtained by subtracting from the rate half its square. The square of .06 is .0036, ½ of which is .0018; .06 — .0018 = .0582.

Or another approximation is by taking the mean between the

interest	.06
and the corresponding discount	.0566
which added together give	.1166
and half of this is	.0583

Annual Interest

The great majority of investments pay interest semi-annually. Occasionally annual-interest securities are offered, and it will be useful to be able to know approximately, for the purpose of comparison with the ordinary semi-annual securities, the equivalent rates. Thus, a 5% income paid annually is worth only as much as a 4.94% income paid semi-annually.

These equivalents are only carried to the nearest one-thousandth of one per cent.

TABLE OF EQUIVALENT RATES OF INTEREST
CONVERTED SEMI-ANNUALLY AND ANNUALLY

2.50%	annual	interest	is	equivalent	to	2.485	semi-annually
2.55%	“	“	“	“	“	2.535	“
2.60%	“	“	“	“	“	2.584	“
2.65%	“	“	“	“	“	2.633	“
2.70%	“	“	“	“	“	2.68	“
2.75%	“	“	“	“	“	2.731	“
2.80%	“	“	“	“	“	2.78	“
2.85%	“	“	“	“	“	2.83	“
2.90%	“	“	“	“	“	2.88	“
2.95%	“	“	“	“	“	2.928	“
3.00%	“	“	“	“	“	2.978	“
3.05%	“	“	‘	“	“	3.027	“
3.10%	“	“	“	“	“	3.076	“
3.15%	“	“	“	“	“	3.126	“
3.20%	“	“	“	“	“	3.174	“
3.25%	“	“	“	“	“	3.224	“
3.30%	“	“	“	“	“	3.273	“
3.35%	“	“	“	“	“	3.322	“
3.40%	“	“	“	“	“	3.37	“
3.45%	“	“	“	“	“	3.42	“
3.50%	“	“	“	“	“	3.47	“
3.55%	“	“	“	“	“	3.518	“
3.60%	“	“	“	“	“	3.568	“
3.65%	“	“	“	“	“	3.618	“

3.70%	annual	interest	is	equivalent	to	3.668	semi-annually
3.75%	"	"	"	"	"	3.718	"
3.80%	"	"	"	"	"	3.765	"
3.85%	"	"	"	"	"	3.814	"
3.90%	"	"	"	"	"	3.862	"
3.95%	"	"	"	"	"	3.912	"
4.00%	"	"	"	"	"	3.96	"
4.05%	"	"	"	"	"	4.01	"
4.10%	"	"	"	"	"	4.058	"
4.15%	"	"	"	"	"	4.108	"
4.20%	"	"	"	"	"	4.157	"
4.25%	"	"	"	"	"	4.206	"
4.30%	"	"	"	"	"	4.255	"
4.35%	"	"	"	"	"	4.304	"
4.40%	"	"	"	"	"	4.352	"
4.45%	"	"	"	"	"	4.402	"
4.50%	"	"	"	"	"	4.45	"
4.55%	"	"	"	"	"	4.50	"
4.60%	"	"	"	"	"	4.549	"
4.65%	"	"	"	"	"	4.597	"
4.70%	"	"	"	"	"	4.646	"
4.75%	"	"	"	"	"	4.695	"
4.80%	"	"	"	"	"	4.744	"
4.85%	"	"	"	"	"	4.792	"
4.90%	"	"	"	"	"	4.842	"
4.95%	"	"	"	"	"	4.89	"
5.00%	"	"	"	"	"	4.94	"
5.25%	"	"	"	"	"	5.184	"
5.50%	"	"	"	"	"	5.428	"
5.75%	"	"	"	"	"	5.67	"
6.00%	"	"	"	"	"	5.91	"
6.25%	"	"	"	"	"	6.156	"
6.50%	"	"	"	"	"	6.398	"
6.75%	"	"	"	"	"	6.64	"
7.00%	"	"	"	"	"	6.882	"

Quarterly Bonds Compared with Semi-annual

Quarterly bonds also occur, but with less frequency than semi-annuals. Some companies, in order to induce holders of bonds to register them, pay interest quarterly after registration, but semi-annually while in coupon form. It is therefore desirable to know approximately how much improvement in income will result from the quarterly payments.

TABLE OF EQUIVALENT RATES OF INTEREST CONVERTED SEMI-ANNUALLY AND QUARTERLY.

2.50%	quarterly	interest	is	equivalent	to	2.508%	semi-annually
2.55%	"	"	"	"	"	2.558%	"
2.60%	"	"	"	"	"	2.608%	"
2.65%	"	"	"	"	"	2.659%	"
2.70%	"	"	"	"	"	2.709%	"
2.75%	"	"	"	"	"	2.76%	"
2.80%	"	"	"	"	"	2.81%	"
2.85%	"	"	"	"	"	2.86%	"
2.90%	"	"	"	"	"	2.91%	"
2.95%	"	"	"	"	"	2.961%	"

3.00%	quarterly	interest	is	equivalent	to	3.011%	semi-annually
3.05%	"	"	"	"	"	3.062%	"
3.10%	"	"	"	"	"	3.112%	"
3.15%	"	"	"	"	"	3.162%	"
3.20%	"	"	"	"	"	3.213%	"
3.25%	"	"	"	"	"	3.263%	"
3.30%	"	"	"	"	"	3.314%	"
3.35%	"	"	"	"	"	3.364%	"
3.40%	"	"	"	"	"	3.414%	"
3.45%	"	"	"	"	"	3.465%	"
3.50%	"	"	"	"	"	3.515%	"
3.55%	"	"	"	"	"	3.566%	"
3.60%	"	"	"	"	"	3.616%	"
3.65%	"	"	"	"	"	3.667%	"
3.70%	"	"	"	"	"	3.717%	"
3.75%	"	"	"	"	"	3.768%	"
3.80%	"	"	"	"	"	3.818%	"
3.85%	"	"	"	"	"	3.868%	"
3.90%	"	"	"	"	"	3.919%	"
3.95%	"	"	"	"	"	3.969%	"
4.00%	"	"	"	"	"	4.02%	"
4.05%	"	"	"	"	"	4.071%	"
4.10%	"	"	"	"	"	4.121%	"
4.15%	"	"	"	"	"	4.172%	"
4.20%	"	"	"	"	"	4.222%	"
4.25%	"	"	"	"	"	4.272%	"
4.30%	"	"	"	"	"	4.323%	"
4.35%	"	"	"	"	"	4.374%	"
4.40%	"	"	"	"	"	4.424%	"
4.45%	"	"	"	"	"	4.475%	"
4.50%	"	"	"	"	"	4.525%	"
4.55%	"	"	"	"	"	4.576%	"
4.60%	"	"	"	"	"	4.626%	"
4.65%	"	"	"	"	"	4.677%	"
4.70%	"	"	"	"	"	4.728%	"
4.75%	"	"	"	"	"	4.778%	"
4.80%	"	"	"	"	"	4.829%	"
4.85%	"	"	"	"	"	4.879%	"
4.90%	"	"	"	"	"	4.93%	"
4.95%	"	"	"	"	"	4.981%	"
5.00%	"	"	"	"	"	5.031%	"

SECTION 8. VALUATION OF BONDS

Article 79, 80

All bonds are supposed to be semi-annual unless otherwise stated.

Difference of Rates

(60) In a 4% bond to net 2½%, what is the difference of rates? (61) In a 3% bond to net 2½%? (62) In a 5% bond to net 3.40%? (63) In a 3% bond to net 3.40%? (64) In a 7% bond to net 4%? (65) In a 5% bond to net 4.80%? (66) In a 3.65% bond to net 5%?

Article 86, 90

Remembering that premium or discount is the present worth of an annuity of the difference of rates and that you have already computed present worths at the foregoing rates (Problems 35 to 39, Section 4), find the premium or discount, and hence the value, of the following, on $1,000:

(67) Value of a 4% bond to net 2½, 15 years.
(68) Value of a 3% bond to net 2½, 15 years.
(69) Value of a 5% bond to net 3.40, 25 years.
(70) Value of a 3% bond to net 3.40, 25 years.
(Discount, not premium).
(71) Value of a 7% bond to net 4, 10 years.
(72) Value of a 5% bond to net 4.80, 34 years.
(73) Value of a 3.65% bond to net 5%, 35 years.

Article 125

Successiv Method

By adding the net income for one period to each of the above values, and subtracting the cash interest, obtain the next periodic value, 14½, 24½, 9½, 33½ and 34½ years. Continue this operation as many times as you please, and at any point prove your work by a fresh computation of the annuity.

(74) Find the value of a 4½% bond for $10,000, 3 years, to net 3½%. From this, work out the values successivly down to par at maturity and construct a schedule as in Art. 94.

(75) Perform the same operation with a 4% bond, a 3% and a 2 % bond.

Article 94
Page 38

The examples in Art. 94 are not worked out in the text. We will give the operation without logarithms and introducing some variations in method.

5% bond for $100,000 to net 4%, 5 years.

Arithmetical Method

The problem is to find the present value of an annuity of $500 for 10 periods at the ratio 1.02; but we require also the separate present worths of the installments.

Present Worth of each Difference

The first installment will be $500 ÷ 1.02^{10}; the second, $500 ÷ 1.02^{9}, etc. As multiplication is easier than division, it will be best to obtain first the value of 1 ÷ 1.02^{10}; 500 times this will be the first installment, or the first entry in the amortisation column.

After multiplying 1.02 by itself, we do not again use it as a multiplier, but square the square, giving the fourth power. The 4th x the 4th gives the 8th and the 8th x the 2d gives 102^{10}:

10th Power of 1.02

102	
102	
102	
204	
10404	(102^2) It is unnecessary to repeat the multiplier.
41616	
41616	
108243216	$(102^4) = (102^2)^2$
865945728	
21648643‖	Contracted Multiplication.
4329729‖	
324730‖	
21649‖	
1082‖	
649‖	
117165938 10‖	$(102)^8 = (102^4)^2$
10404	
117165938 10‖	
468663752‖	
4686638‖	
121899442 00	$(102)^{10} = (102^8) \times (102^2)$

Tabular Multiplication

Next, 1 is to be divided by 1.21899442. We shall use contracted multiplication and further facilitate the work by employing the *Tabular Plan.* This consists in preparing in advance a table of the first 9 multiples of 1.21899442 in such a way that we are certain of their correctness.

On the first line of the table we set down the number, and on the second line, its double.

1	1 2 1 8 9 9 4 4 2
2	2 4 3 7 9 8 8 8 4
3	
4	
5	
6	
7	
8	
9	
Proof	

The third line is formed by adding the first to the second, and all the others in succession by adding the first. The proof line is 10 times the original, if there is no mistake in the work.

1	121899442
2	243798884
3	365698326
4	487597768
5	609497210
6	731396652
7	853296094
8	975195536
9	1097094978
Proof	1218994420

Contracted Division

The contracted division consists in merely subtracting these multiples. The quotient may as well be placed *above* the dividend to save space.

Quotient	820348300
Dividend	1000000000
(8)	975195536
	24804464
(2)	24379888‖
	424576
(03)	365698‖
	58878
(4)	48760‖
	10118
(8)	9752‖
	366
(3)	366‖

.8203483 is, therefore, the present worth of $1 due in 5 years; its product by 500 is the first amortisation

	410.17415
Subtracting this from	500.
gives the compound discount	89.82585
Dividing this by .02 gives	4491.2925
which is the premium, rounded to	4491.29‖

Formation of Successiv Amortisations

Our amortisation column will begin with $410.17, and each successive term will be 1.02 times the preceding, while the sum of the column must be 4491.29. To insure accuracy in the last figure, it will be well to retain at least the mills. Having obtained all the terms, the operation is performed once more, giving as a test $500. The terms are again tested by addition, bringing the result, 4491.29. Then the book values beginning with $104,491.29, and ending with $100,000, are formed by subtraction, still retaining the mills. In making up the schedule the values are rounded to the nearest cent and the amortisation column made to correspond.

410.174	104,491.292	410.174
418.377	104,081.118	418.377
426.745	103,662.741	426.745
435.280	103,235.996	435.280
443.986	102,800.716	443.986
452.866	102,356.730	452.866
461.923	101,903.864	461.923
471.161	101,441.941	471.161
480.584	100,970.780	480.584
*490.196	100,490.196	490.196
Total, 4,491.292	100,000.000	

[*490.196 × 1 02 = 500]

Test by Differencing

Growing Errors

In a successiv computation like the one just given, a slight error increases at every step, and there is danger that a great many terms may have to be recalculated. The method of *differencing,* applied during the progress of the work, will form an efficient check on all except the last figure.

Successiv Columns

To difference a series, we first set down its terms in a first column. In the second column we set down the first differences (D_1), of which the first line is the difference between the first term and the second, the second line is the difference between the second and the third, and so on. D_2 is composed of the differences between these differences. D_3 is formed from D_2 in just the same way as D_2 from D_1, and all succeeding differences in the same way, to the extent required.

Article 94

The terms just obtained in amortising 104,491.292 down to par, would be differenced as follows:

Term	D_1	D_2	D_3
410.174	8.203	.165	.002
418.377	8.368	.167	.004
426.745	8.535	.171	.003
435.280	8.706	.174	.003
443.986	8.880	.177	.004
452.866	9.057	.181	.004
461.923	9.238	.185	.004
471.161	9.423	.189	.003
480.584	9.612	.192	
490.196	9.804		
500.000			

Intentional Errors

To appreciate the utility of the method, you should introduce an error purposely by altering one of the figures in a term at least three or four lines from the top. Even a mill, when all the differences are carried out, will show violent fluctuations in the column D_3 and instantly call attention to the error.

Rejected Decimals

The reason why the third column shows some fluctuation even though no errors have been made, is that the terminal figure of the terms is never accurate, but always rounded off or up. In a third difference-column this residue of error increases threefold; in a fourth column it may reach six times the original rounding, and in the fifth, ten times.

Limit of Tolerance

The extent to which the last column of difference may be allowed to "waver" will be learned by experience. The next-to-the-last column should be progressiv; that is, it should never change its course and go backward.

Test Exercises

It will be a useful exercise to take the more extended value 410.17415 (instead of 410.174), multiply it up to 500, and difference the results out to 5 differences. A very minute error will become enormously magnified and call attention to itself.

SOLUTIONS: (60) 0.75%. (61) 0.25%. (62) 0.8%. (63) 0.2%. (64) 1.5%. (65) 0.1%. (66) 0.675%. (67) 1,186.67. (68) 1,062.22. (69) 1,268.01. (70) 933.00. (71) 1,245.27 (72) 1,033.36. (73) 777.94. (74) 10,282.45. (75) 10,141.22; 9,858.78; 9,576.33.

SECTION 9. BROKEN INITIAL AND SHORT TERMINAL

Articles 97, 98

(76) Suppose the value of a 4% bond for 15 years on a 2½% basis to be (as shown in problem 67) 1.18666680; what would be its value one month later, the time then being 14 years 11 months? As we are dealing with half years, this time must be treated as 14½ years 5 months, or 15 years less 1/6 of the amortisation period.

The theoretical or mathematically correct value (Art. 98) in the above case would be ascertained as follows:

The ratio of increase is 1.0125
Its logarithm is.. .005 395 031 887
This must be divided by 6, giving...................... .000 899 171 981
which is the logarithm of the effectiv ratio for 1-6 of a period equivalent to 1.0125 for a period.
The corresponding number is.......................... 1.002 072 564 8
Multiplying by this 1.18666680, the value at the beginning and we have the flat value....................... 1.189 126 21

Although never used in actual buying and selling, yet this method is proper for estimating results of financial operations.

(77) A firm of brokers makes this offering:
$50,000 3% bonds of the .... of ...., due July 1, 1921, J. & J. on a 2½% basis. What should be the price on September 25, 1906, flat or "and interest?"

(78) $25,000 5% bonds due April 1, 1930, A. & O., to yield 3.40%. What is the price flat on July 10, 1905? What is the price "and interest"?

(79) $10,000 3% bonds due January 1, 1930, J. & J. to net 3.40%. Price exclusiv of interest and price flat, on May 16, 1905.

Short Terminal

(80) $6,000 4½% bonds issued in 1900 due April 1, 1920, M. & N., on July 1, 1906, on 4^{80} basis.

(81) An investor owns the 4 lots of bonds mentioned in Nos. (77) (78) (79) and (80) and has hitherto carried them on his books at par. He desires to have them adjusted to investment value as of December 31, 1906. What will be the investment value of each lot? of the aggregate?

(82) Find the amount of amortisation for the *next* half-year (January-June, 1907), on each of these bonds and the total amortisation for the half-year.

(83) Taking the bonds in Problem 78, ascertain their values on April 1 and October 1, 1929, and thence on July 1, 1929. Amortise to January 1, 1930, and then for the broken period to April 1, 1930, when they should reduce to par. **Articles 103**

(84) Taking the bonds in Schedule I., Art. 112, reconstruct the schedule so that the next date after May 1, 1904, is July 1; then January 1, 1905, and so on at balancing periods, giving a J. & J. schedule instead of M. & N. The short terminal July, October, 1909, being also a broken period is peculiar. **Articles 101, 102, 111, 112**

(85) A certain issue of $100,000 4% bonds is dated September 1, 1905, and interest begins at that date; but interest is payable on February 1 and August 1, and the principal (with 4 months' interest) is payable December 1, 1909. What is the value of the bonds on a 3.60 basis at that date of issue? What is their value at the same basis if purchased on December 1, 1905? Note that you are interpolating into a 5-month period, not a 6-month, in the beginning.

(86) Make a schedule running from December 1, 1905, to maturity of the above bonds with F. & A. dates.

(87) Make a schedule as above, but with J. & J. dates, for balancing purposes.

(88) A broker offers you the above bonds on December 1, 1905, at 101.50 (meaning $101.50 for each $100, which is the customary phrase) which he says will pay *about* 3.60. Eliminate any residue by the methods in Articles 107-110, making a J. & J. schedule running to maturity. This example contains all the following peculiarities: Short initial period, odd purchase date in that period, short terminal period, interpolated balance dates and residue to be eliminated.

SOLUTIONS: (76) 1.18913902.

(77) 53,420.93 flat. (78) 31,996.64 flat, or 31,652.89 and interest. (79) 9,336.43 and interest, or 9,448.93 flat. (80) 5,820.34 and interest. (81) 53,025.02 31,392.26, 9,365.30, 5,825.03, 99,607.59 (82) Amortisation, 87.19, 91.33; accumu-

lation, 9.21, 4.80; net reduction, 164.51. (83) Jan. 1, '30 25,098.33; interest on premium, 1.67; interest on par, 212.50; interest collected, 312.50. (84) July, '06, 100,245.92. (85) September 1, '05, 101,563.90; December 1, '05, 101,477.98; August 1, '09, 100,131.75. (86) February 1, '06, 101,420.69. (87) January 1, '06, 101,449.33. July, '06, 101,275.34 (This as well as the preceding value must be inserted by interpolation, not by derivation.) (88) Residue, 22.02; co-efficient for elimination, 1.0148987.

SECTION 10. DISCOUNTING

Article 115

This discounting process may be performed as well by multiplication as by division, and multiplication is preferable as the more direct and compact process. In Table VI are the reciprocals of all usual ratios of increase. Multiplying by .9803921568, for example, will give the same result up to a certain number of places, as dividing by 1.02. Using the tabular plan, we have this table:

Table of Multiples

1		9	8	0	3	9	2	1	6
2	1	9	6	0	7	8	4	3	1
3	2	9	4	1	1	7	6	4	7
4	3	9	2	1	5	6	8	6	3
5	4	9	0	1	9	6	0	7	8
6	5	8	8	2	3	5	2	9	4
7	6	8	6	2	7	4	5	1	0
8	7	8	4	3	1	3	7	2	6
9	8	8	2	3	5	2	9	4	1

In our example of a 5% bond, yielding 4%, we should proceed thus:

102,500.00
98,039.22
1,960.78
490.20
100,490.20
2,500.00
102,990.20
98,039.22
1,960.78
882.35
88.24
.20
100,970.79

Brought forward	100,970.79
	2,500.00
	103,470.79
	98,039.22
	2,941.18 etc.

There is an error of 1 cent in the value 100,970.79; this could easily have been prevented by carrying out into mills. For long operations it is always advantageous to use a few spare places beyond those retained in the final result.

Article 94
Present Worths of Interest-Difference

Still greater brevity will be attained by working out first the items of amortisation, or present worths of the difference of interest. The present worth of the interest-difference 500 are obtained as follows, using fewer figures and less labor than in the preceding example:

500.000	
490.196	½ year before maturity
392.157	
88.235	
.098	
.088	
6	
480.584	1 year before maturity
392.157	
78.431	
.490	
.078	
4	
471.160	1½ years before maturity
392.157	
68.627	
.980	
98	
59	
461.921	2 years before maturity

Writing these down in reverse order, the amortisation column of the schedule is filled:

461.92
471.16
480.58
490.20

1903.86

The value at two years is, therefore, 101,903.86 and the schedule may be further filled:

Amortisation	Book Value	Par Value
	101,903.86	100,000
461.92	101,441.94	
471.16	100,970.78	
480.58	100,490.20	
490.20	100,000.00	

For practice, any of the problems (77) to (80) may be worked over backwards.

Discounts from Table II

If the rate is one of those embraced in Table II and the difference of interest is a simple number, the process is still easier. Here the present worths of 500 for various numbers of periods at 2% per period are required. In Table II. we find these present worths for $1; pointing off 3 places to the right gives the corresponding values for $1,000, and halving this, all in the one operation gives the successiv figures required:

4 periods .92384543 × 1000 ÷ 2 = 461.9227‖
3 " .94232233 " 471.1617‖
2 " .96116878 " 480.5843‖
1 " .98039216 " 490.1961‖

1903.8648‖

Article 116

Reussner's Tables

Mr. R. Reussner's "True Discount Tables," recently published, gives multipliers for each day, from 1 to 180, carried to 8 places, for a great number of usual rates, and will much facilitate discounting for fractional periods. In the example in the text it gives .99009901 opposite 90 days at 4%, with the following result:

102,500.000

99,009.901
1,980.198
495.050

101,485.149, same as in the text.

SECTION 11. SERIAL BONDS

Article 117

A city issues 10 4% bonds for $10,000 each, A. & O., on April 1, 1900, maturing as follows: $10,000 on April 1, 1902; $10,000 on April 1, 1904, and so on, $10,000 each alternate year, the last $10,000 on April 1, 1920. They are sold at 108.33, the purchaser believing that he has a 3.10% investment. How near right is he?

"Average Maturities"

As the average time of the bonds is 11 years, it might be inferred that the true value of the series was the value of a single bond of $100,000 due in 1911, which would be $108,334.54; but this is fallacious; the true price, obtained by adding together all the separate tabular values is always less.

Separate Maturities

The bond due in	1902	is worth	10,173.24
	1904		10,336.13
	1906		10,489.31
	1908		10,633.35
	1910		10,768.79
	1912		10,896.16
	1914		11,015.92
	1916		11,128.53
	1918		11,234.43
	1920		11,334.01
	Total series		108,009.87

It is evident that the purchaser should have paid 108.01 instead of 108.33, and that on the latter price he will earn less than 3.10. How much less, is to be ascertained.

The value at 3.10 might have been carried out further in decimals to the limit of the tables, giving 108,009.8686.

The values at 3.05 will next be copied down, and as the result is too large, showing that the rate lies between, we copy also the tabular differences and sub-differences from the blue pages of the bond tables.

			Values	Diff.	Sub.-Diff.
3.05	2	years	10,182.9714	.0009	
	4	"	10,355.1945	.0033	
	6	"	10,517.3006	.0069	
	8	"	10,669.8840	.0117	.0001
	10	"	10,813.5041	.0174	.0001
	12	"	10,948.6875	.0240	.0001
	14	"	11,075.9297	.0314	.0002
	16	"	11,195.6973	.0395	.0003
	18	"	11,308.4294	.0482	.0004
	20	"	11,414.5391	.0574	.0006
			108,482.1376	.2407	.0018

Inter-rates

The inter-rates, 3.06, 3.07, 3.08 and 3.09 can now be obtained in bulk without determining the values for separate years, according to the directions on page 123 on the bond tables.

Find the difference between	108,482.1376
and	108,009.8686
which is	472.2690
⅕ of this is	94.4538

Subtracting from 108,482.1376 successively ⅕; ⅖; ⅗; ⅘ we have the approximate values	
for 3.06	108,387.6838
for 3.07	108,293.2300

But it is unnecessary to go further; it is evident that the effective rate is a little below 3.07.

Subtracting 1½ times the tabular difference (.2407), and adding 1/10 the sub-difference (.0018), we have the corrected value at 3.07, viz.:

108,293.2300 — .3611 + .0002 =	108,292.8691
with a residue to be eliminated	37.1309
making the price paid	108,330

Successiv Method

The values at 3.07 must next be worked out for each period down to the last maturity.

April 1, 1900,	108,292.8691
× 1.01535	1,082.9287
	541.4643
	32.4879
	5.4146
—	2,
Oct. 1, 1900,	107,955.1646
	1,079.5516
	539.7758
	32.3865
	5.3978
—	2,
April 1, 1901,	$107,612.2763

Balancing Period

But it may be that balancing-period figures are wanted, say at J. & J. In that case, the value on July 1, 1900, is halfway

between	108,292.8691
and	107,955.1646
or	108,124.0168
with which we continue	1,081.2402
	540.6201
	32.4372
	5.4062
—	2,
January, 1901,	107,783.7205
	1,077.8372
	538.9186
	32.3351
	5.3892
—	2,
July 1, 1901,	107,438.2006
	1,074.3820
	537.1910
	32.2315
	5.3719
—	2,
Jan. 1, 1902,	107,087.3770

First Payment in Series

We have now reached a point where a broken terminal period occurs, as to the first $10,000 due April 1, 1902, and we must follow the directions of article 103, with this modification, that the $10,000 and the remaining $97,087.38 must be treated separately.

	107,087.3770
Amount of principal due April 1,	10,000.
Remainder	97,087.3770
The usual procedure	
	970.8738
	485.4369
	29.1262
	4.8544
10,000 × .007675 (3 months)	76.7500
	98,654.4183
Income 2% on $90,000	
1% " $10,000 =	1,900.
July 1, 1902,	96,754.4183

This will exemplify the process when one portion of serial bonds is paid off.

Elimination of Residue
Article 110

There is a residue of $37.1309 to be eliminated, for which we shall use the third method. A total premium of $8,330 is to be amortised, while the 3.07 basis will amortise only $8,292.8691. The proportion is 8330 ÷ 8292.8691 = 1.0044784. A table formed from this will give the following multiples:

1	1004478
2	2008957
3	3013435
4	4017914
5	5022392
6	6026870
7	7031349
8	8035827
9	9040306

The amortisation at 3.07 for the fractional period and the 4 full periods is as follows:

168.8523
340.2963
345.5199
350.8236
332.9587

and as adjusted for elimination

Eliminands	1688523	3402963	3455199	3508236	3329587
	1004478	3013435	3013435	3013435	3013435
	602687	401791	401791	502239	301344
	80358	2009	50224	8036	20090
	8036	904	5022	201	9040
	502	63	199	36	502
	23				87
Eliminates	1696084	3418202	3470671	3523947	3344498

The schedule will then be made up as follows to this point:

Schedule

Date	Total Int. 4%	Net Income 307+%	Amortisation	Book Value	Par
1900 April 1,	Cost			108,330.00	100,000
July 1,	1000	830 39	169 61	108,160.39	
1901 Jan. 1,	2000	1658.18	341 82	107,818.57	
July 1,	2000	1652 93	317.07	107,471.50	
1902 Jan. 1,	2000	1647.61	352.39	107,119 11	
July 1,	1900	1565.55	335.45	96,784.66	90,000

The premium is now $6,784.66, and the premium at 3.07 is $6,754.42, which we test as follows:

675442

602687
70313
5022
402
40
2

Proof 678466

Uneven Loans

The terms of a series of bonds need not necessarily be of like amount. Suppose the payments in the above example were

$10,000	in 1902
$20,000	in 1904
$30,000	in 1906
$40,000	in 1908
$100,000	

and it were desired to find the value at 3.10; the process would be

10,173.2358	10,173.2358
10,336.1340 × 2	20,672.2680
10,489.3124 × 3	31,467.9372
10,633.3506 × 4	42,533.4024
Value of series	104,846.8434

The formation of the schedule would be precisely analogous to that already given.

Tabular Methods

Most serial bonds run by years an even amount annually. Where the rate is one ending in 5 or 0, and the values for exact interest periods are required, not for intermediate periods, a simpler process may be used, copying values direct from the tables. A series of 10 4% bonds, J. & J. issued July 1, 1900, payable $1,000 each first of July, 1901 to 1905, is sold on a 3.50 basis. Set down in two columns, the first ten values from the tables; add and subtract successivly.

½ y.	1002.457	1 y.	1004.872	
1½	1007.245	2	1009.577	
2½	1011.870	3	1014.122	
3½	1016.337	4	1018.513	
4½	1020.651	5	1022.753	
			5069.837	July, '00
Jan., '01	5058.560			
			1022.753	
			4047.084	July, '01
	1020.651			
Jan., '02,	4037.909			
			1018.513	
			3028.571	July, '02
	1016.337			
Jan., '03	3021.572			
			1014.122	
			2014.449	July, '03
	1011.870			
Jan., '04,	2009.702			
			1009.577	
			1004.872	July, '04
	1007.245			
Jan., '05,	1002.457			

Formula for Serials

The total value of an annual series may be obtained by the following formula:

Let m be the number of different maturities and n the number of the periods the last bond has to run. Let r (for brevity) represent the ratio of increase, instead of $1 + i$. The powers of r are obtainable from Table I or by logarithms. The principal of each bond being $1, the formula would read:

$$\text{Aggregate premium} = \left(m - \frac{r^{2m}-1}{(r^2-1)r^n}\right) \div i \times (c-i)$$

In the preceding example $m = 5$, $n = 10$, $r = 1.0175$, $i = .0175$, $c = .02$, $c-i = .0025$.

From Table I, or from the Extended Bond Tables:

$$\begin{array}{llcr}
 & r^2 = & & 1.03530625 \\
 & r^{2m} = & r^{10} = & 1.18944449 \\
 & r^n = & r^{10} = & 1.18944449 \\
\text{Therefore:} & r^{2m}-1 = & = & .18944449 \\
 & r^2-1 & = & .03530625 \\
 & \dfrac{r^m-1}{(r^2-1)r^{10}} & = & \dfrac{.18944449}{.03530625 \times 1.18944449} \\
 & & = & 4.511139
\end{array}$$

$$m - \frac{r^{2m}-1}{(r^2-1)r^n} = 5 - 4.511139 = .488861$$

$$\left(m - \frac{r^{2m}-1}{(r^2-1)r^n}\right) \div i = 488861 \div .0175 = 27.93491$$

$$\text{Value of series} = 5 + 27.93491 \times .0025 = 5.0698373,$$

which is the result already obtained by addition.

This formula will seldom be of use unless in case of a very complex rate not comprised in the tables. It will then involve the computation of three powers of r by logarithms. It is now publisht for the first time.

The following problems may be solved in either way:

(89) A corporation has issued a series of 10 \$1,000 bonds, 5%, M. & N., on May 1, 1901; payable each May 1, 1909 to 1918. What is the value on a 3.60 basis, May 1, 1906? on July 1, 1906? on August 23, 1906?

(90) Find the values as above, but on a 4% basis.

SOLUTIONS: (89) May 1, 1906, 10,897.40; July 1, 1906, 10,962.79 flat; August 23, 11,019.45 flat. (90) 10,630.42; 10,701.29 flat; 10,762.71 flat.

Bonds purchased flat should always be separated into principal and interest before entering them in the books.

SECTION 12. OPTION OF REDEMPTION.

The rate of income on a bond subject to a right to redeem at an earlier date than that of actual maturity and upon payment of a premium, can be ascertained by means of tables. Only the income which is certain must be calculated upon in advance; hence there will always be a contingent profit which may be realized. **Article 127**

For example suppose the bond to be a $4\frac{1}{2}$% one absolutely due in 30 years but redeemable at 105 after 20 years; issued 1905, redeemable 1925, payable 1935.

Neutral Rate

First; To ascertain the limit between the cases where the redemption is a benefit and where it is a disadvantage. For this purpose, we must suppose ourselves to be in 1925 at the redemption date. This bond now has 10 years to run. Turning to the 4½% bond table, under 10 years, we find that 1.05 is the price almost exactly at a 3.89 basis. Therefore, if the bond is bought now on a 3.89 basis, the investment value in 1925 will be exactly 1.05 and there will be neither profit and loss in being required to surrender at 1.05.

3.89% may be called the neutral rate.

Advantageous Redemption Ignored

Bearing in mind that the higher the rate of interest the lower the premium: If the rate be more than 3.89 say 4% the option may be disregarded, for we shall surely have 4% for 20 years, and probably for the full time. In case the rate of interest has fallen to 3.89 the issuer of the bond may think it advantageous to redeem, so as to sell his new issue at more than .05 premium. Then as our bond stands at less than 1.05 we get a profit besides our 4% income.

Thus, if the bond is bought at a basis which yields more than 3.89 for 30 years, we may safely amortise at that basis for 20 years, or until the option is exercised.

Disadvantageous Redemption Expected

But if the rate for thirty years, which we may call the apparent rate, or non-redemption rate, is less than 3.89, the bond will be worth more than 105 at the redemption date and the issuer may be expected to redeem. If he does not, it is because the general rate of interest has risen so that he must pay more than 3.89% in which case he will allow us to continue at 3.89 till maturity.

Thus, if the bond is bought at a price which would be an apparent basis of less than 3.89, redemption, being adverse to our interests, must be expected.

The redemption date is then the actual maturity, but the principal is not 1 but 1.05.

Change in Principal

Let the par be 100,000 and the price $114,423.38 which is at the apparent basis 3.70. To get the actual basis we must consider the par as 105,000 and the time 20 years. But if the par is 105,000, the cost is not at 1.1442|| but at 1.1442||÷1.05 =1.0897||. The cash rate is also transformed; the cash income is still $4,500, but this is not 4½% of $105,000; it is only 4 2-7%.

Therefore the limitation imposed by the option of redemption entirely changes the problem. Instead of a 4½% bond for $100,000 due in 30 years bought at 1.1442 we have a 4 2-7% bond for 105,000 due in 20 years bought at 1.0897.

No tables have been publisht for 4 2-7% bonds; if this exact case of 4½% bonds redeemable at 1.05 were very frequent such tables would doubtless be publisht. However we can easily construct them by adding to the value of a 4% bond, 2-7 of the difference between a 4 and a 5.

Approximate Location

As a rough approximation, find 1.0897 as closely as possible in the 20-year tables for 4% and 5% respectively. The nearest to 1.0897 in the 4% table is 1.08655516, which is a 3.40 income; the nearest in the 5% table is 1.08623676, 4.35 income. The required rate will be about 2-7 of the distance between 3.40 and 4.35.

4.35—3.40= .95
2-7 of .95 = .27
3.40+.27= 3.67

Therefore 3.67 is the approximate rate, and we might begin testing with that rate. We notice, however, that the approximations 1.08656|| and 1.08624|| are both short of 1.0897; hence probably the rate will fall short of 3.67, and it will be easier to start with the tabular rate 3.65. In fact, had we gone a little further in decimals, we should have obtained 4.33 and 3.38; and 2-7 of the difference would have located the rate as 3.65.

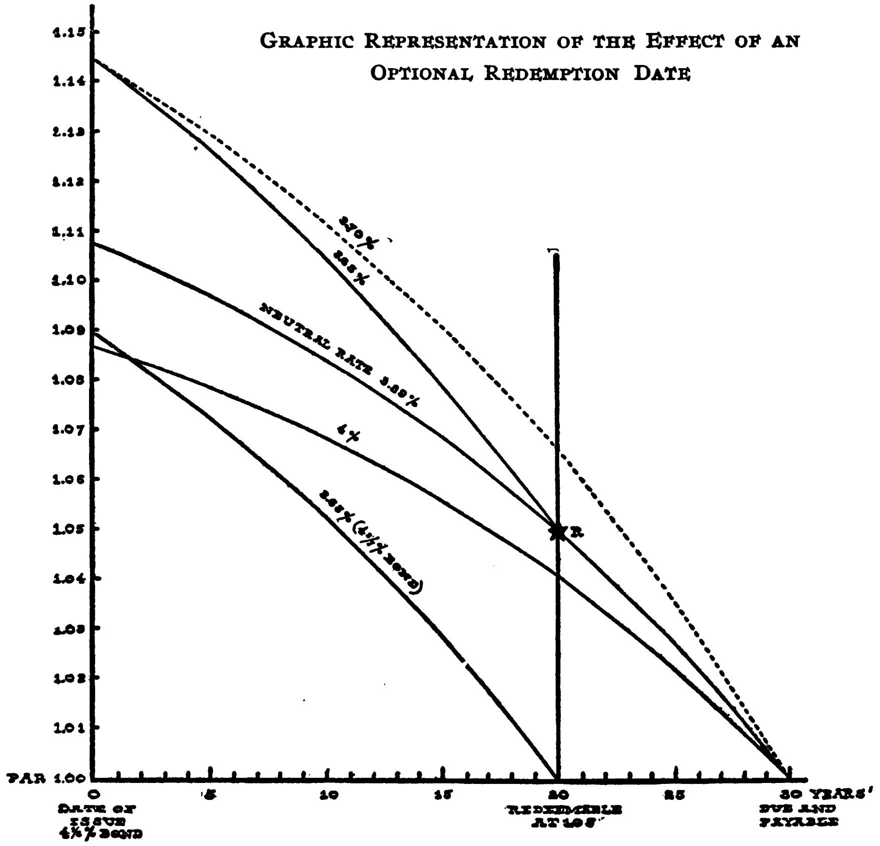
4% table 20 years	3.65	1.0493748
5% " " "	3.65	1.1904458
Difference		.1410710
1-7		.0201530
2-7		.0403060
Add to 4% value		1.0493748
4 2-7% "		1.0896808

This value is very close to 1.0897.

Value of $105,000 at the same price	114416.48
Actual price	114423.38
Residue	6.90

This is the nearest approximation we can obtain without using more decimals; therefore 3.65 is the actual rate of income for a 4½% bond redeemable at 1.05 10 years before maturity, if purchased at 114.42, 30 years before maturity.

In the following diagram, the dotted line marked 3.70 is the apparent course of a bond at 114.42, 30 years to run; but the option at 105 pulls it down to a 3.65 basis; during the last 10 years it earns 3.89 if not redeemed. The 4% line, as it passes below the 105 point, is unaffected.


GRAPHIC REPRESENTATION OF THE EFFECT OF AN OPTIONAL REDEMPTION DATE

Completion of Schedule

To complete a schedule running from the date of issue to that of redemption, we have the following data:

Par $105,000.

Cash interest, semi-annually, $2,250, being at the rate of 4 2-7% per annum.

Net income, semi-annually, $1,916.25, being 3.65% per annum on $105,000.

Difference of interest, $2,250 — 1,916.25 = 333.75.

Present worth of 20-year annuity of $333.75 each half year, $9,416.48. Present value of bond at 3.65%, $114,416.48. Actual value, $114,423.38. Eliminand, $6.90.

We might now proceed to amortise $114,416.48 down to maturity. Each term would then have to be corrected to eliminate the residue, $6.90. The multiplier for this purpose would be.

9423.38 ÷ 9416.48 = 1.00073276||

Discounting Process

But we may proceed in the other direction and discount $333.75 at various dates as in Section 8; this has the great advantage that $333.75 may be first × 1.00073276, thus accomplishing the elimination process once for all.

333.75 × 1.00073276 = 333.99456;

This last is substituted as a base in place of $333.75, and we proceed to discount using the factor .9820770093|| in the tabular method:

```
333.9946
--------
294.6231
 29.4623
  2.9462
    8839
     884
      39
       6
--------
328.0084 ½ year before maturity
--------
294.6231
 19.6415
  7.8566
      79
       4
--------
322.1295 1 year before maturity
--------
  etc.
```

(91) If a 4% bond is redeemable 25 years before maturity at 105, what is the neutral rate of income?

(92) If a bond reads at 4%, but the amount which will be received is 1.05 of the nominal par, what is the actual percentage of cash income?

(93) A 50-year 4% bond is redeemable at 105 after 25 years. Find its actual income rate if bought at 105, 106, 107, 108 and 109.

(94) A 30-year 5% bond is redeemable at 110 after 15 years. Find at what price it should be bought when issued to pay 3.90? 440?

We are now prepared to formulate a rule for determining the net income yielded at a certain price, by a bond bearing a certain par interest but subject to redemption at another price, on the assumption that the right will be exercised.

E. g.
4½%
÷
1.05
=
4²⁄₇%

1. Divide the nominal rate of interest by the redemption price per unit; the quotient will be the actual cash rate, consisting of a whole number and a fraction.

1.1442
÷
1.05
=
1.0897

2. Divide the purchase price by the same divisor, giving the actual purchase price per unit.

4%
and
5%
20 years

3. Select two different bond-tables, one at a lower, one at a higher cash rate than the actual rate obtained in paragraph 1. These should be even rates, not fractional and 1% apart. Find the column for the number of years before redemption.

Nearest to
1.0897
4% table,
1.08656
5% table,
1.08624

4. In each of these columns find the nearest price to the actual purchase price in paragraph 2.

4.35 and 3.40
Difference .95

5. Set down the two rates of net income found opposit these values, and find their difference.

²⁄₇ of .95
= .27
.27+3.40=
3.67

6. Take such a fraction of the difference as the fractional part of the mixt number which represents the rate; add the result to the smaller rate and the sum is, approximately, the desired yield.

3.65
produces
1.08968

7. Try the nearest rates from the table until one is found which produces the desired price.

SOLUTIONS: (91) 3.69 + (92) 3 17-21, or 3.80952 + (93) Between 3.77 and 3.78; 3.73 +; 3.69; 363 for 25 years. (94) 118.005676; 109.042757.

SECTION 13. HALF-YEARLY INCOME ON QUARTERLY BONDS.

Quarterly-Semi-annual Basis

It is easy to find the value of a quarterly-interest bearing bond to yield a certain quarterly income; but it is a more complicated problem to find the value when the net income of the same bond is to be computed on a semi-annual basis; yet we must be prepared to do this, for the "quarterly-quarterly" value would be of no utility for comparison with the ordinary semi-annual bond.

To demonstrate the process we take the following example:

A bond for $1,000,000 due January 1, 1907, bears interest at 6 per cent. per annum payable quarterly, J A J O, $15,000 each quarter. What is its value on July 1, 1906, on a basis to net 2½ per cent. per annum, payable semi-annually, that is 1¼ per cent. each half year?

If the cash interest on the bond were payable semi-annually, then by the ordinary methods, the value on July 1, would be $1,017,283.95. But the value is considerably enhanced by the fact that, instead of receiving $30,000 on January 1, $15,000 of it will be collectible October 1.

Application of Table

By the rule on page 54, we must take the premium $17,283.95 and multiply it by .0053405, to find the increase of premium caused by the quarterly feature.

$17,283.95 × .0053405 = $ 92.31
1,000,000.00 + 17,283.95 + 92.31 = 1,017,376.26,

which is the value of the quarterly bond, as deduced from the tables.

To test this let us consider the problem by the discount method.

Discount Method

The amount to be received on Jan. 1, 1907, principal and interest, is	$1,015,000.00
The present worth of this on July 1, as obtained by dividing by 1.0125, is	1,002,469.14
The $15,000 to be received on October 1, is also to be discounted, but only for 3 months. If we were computing on a quarterly basis, we should divide by 1.00625, but this would not be quite correct. Our divisor must be the effectiv quarterly equivalent of a semi-annual 1.0125, or 1.00623061.	
15,000 ÷ 1.00623061	14,907.12
Giving the value	$1,017,376.26

As this result agrees with that given by multiplying by .0053405, our method is, so far, corroborated.

Taking this value as ascertained on July 1........$1,017,376.26
and adding to it the interest then due............. 15,000.00

We have the entire value July 1 as...............$1,032,376.26
Placing ourselves now at January 1, the present value of the above sum is..................$1,019,630.87
and that of $15,000 (April 1) is.................. 14,907.12

making the value January 1.....................$1,034,537.99

If our method is correct, it should give the same result. We find that the ordinary semi-annual bond would be worth on January 1 ....$1,034,354.52
Multiplying the premium by the tabular multiplier;
34,354.52 × .0053405 = ... 183.47

and adding it to the value, we corroborate the previous result $1,034,537.99

It is evident that the holder will at maturity receive his principal and will also have had interest at the rate of .0125 every half year on every dollar for the length of time it was invested. For, the amount invested on July 1 being $1,017,376.26, we may divide this into two parts:

$ 14,907.12 to be repaid Oct. 1.
1,002,469.14 to be repaid Jan. 1.

$ 14,907.12 on Oct. 1 has earned
92.88 interest for 3 months.

$ 15,000.00 received.

$1,002,469.14 on Jan. 1 has earned
12,530.86 interest for 6 months.

Holder Receives Just Amount

$1,015,000.00 received.

At the beginning of every half year there is a similar portion ($14,907.12) of the investment which is to receive interest for three months and all the remainder for six months.

Re-Investment Method

We may arrive at the same result in still another way, by considering the October payment, not as written off, but re-invested till January 1 at the same equivalent rate.

Value July 1, 1906		$1,017,376.26
On this there should be earned for the entire half year at .0125		12,717.20
His due on July 1 is		$1,030,093.46
He has received on Oct. 1	$ 15,000.00	
Interest on reinvestment (.00623061) to Jan. 1, 1907	93.46	
On Jan. 1, Cash	15,000.00	
and	1,000,000.00	$1,030,093.46

Here $93.46 is considered as added to the cash-interest instead of its worth being taken from the earnings. In this point of view, the cash interest is actually................$30,093.46
and the income.................................. 12,500.00

hence the difference of rates amounts to.............$17,593.46
instead of 17,500.00

Since each premium is an annuity whose rent is the difference of rates, the premiums on a quarterly conversion must be to those on a semi-annual basis as 17,593.46 to 17,500 or as 1.759346 to 1.75.

$1.759346 \div 1.75 = 1.0053405$.

The figures in Appendix II are obtained by a calculation substantially as above shown.

If instead of .00623061 the conventional rate .00625 had been used, the discrepancy would at first have been only a few cents, but would have increased with the length of time.

Quarterly-Quarterly

The value of a 6 per cent. quarterly bond to net 2½ per cent. quarterly (equivalent to 2.51 semi-annually) would be 1,034,459.89 at one year from maturity, or 1,017,337.29 at a half year. Comparing this with the true value as computed, would show the uselessness of trying to draw any conclusions unless a uniform unit of time is assumed, except on small amounts of principal and short times.

(95) A 5% quarterly bond for $100,000, 5 years to run on a 4% semi-annual basis, what is its value?

(96) Ascertain the value of the same bond at 4½ years.

(97) Derive the 4½ years value from the 5 years, and obtain the same value as in 96.

(98) Find the value of a 2% quarterly bond, 5 years to run to net 1.80% semi-annually.

(99) Two issues of 3½% bonds each $100,000 due in 20 years are offered for sale; one with interest payable semi-annually at 95.29, the other quarterly at 95.38. Which is the better purchase?

(100) Which is the better purchase:
1,000,000 4% quarterly bonds 10 years at 104.33, or
1,000,000 3% semi-annual bonds 10 years at 95.50?

SOLUTIONS: (95) 104,603.02. (96) 104,182.64.

(97) Value at 5 years	$104,603.02
Of this $1,250 is payable in three months.	
Present worth at 4% semi-annually	1,237.69
The remainder	$103,365.33
Produces income at .02	2,067.31
	$105,432.64
Cash interest received	1,250.00
Value 4½ years	$104,182.64

(98) 100,973.61. (99) The quarterly bonds.
(100) The semi-annual bonds.

SECTION 14. BONDS WITH ANNUAL INTEREST.

Annual Coupons Semi-Annual Basis

Bonds, the interest on which is only paid once a year, are somewhat rarer than those where it is paid four times a year; but when they do occur means should be provided for ascertaining the value at any given rate reduced to the standard of semi-annual payment. This is somewhat easier than reducing quarterly interest to semi-annual income.

Discounting

Article 15

We may begin by a simple example using the discount method, either by division or by multiplication. A 4% annual bond, to yield 3% semi-annually, 2 years to run, for $100,000.

Beginning at maturity at par	$100,000.00
and adding to it the annual coupon then due	4,000.00
	$104,000.00
We discount this by dividing by the ratio, 1.015, or, what is the same thing, multiplying by its reciprocal, .98522167; 104,000 ÷ 1.015 or × .98522167 =	102,463.05
This is the value, flat, 6 months before maturity. If there were a payment of interest at this date we should add its value. But there is none, hence we continue the process; 102,463.05 ÷ 1.015 or × .98522167 =	100,948.82
Here we add the coupon payable one year before maturity	4,000.00
	$104,948.82

We discount this for another half year 104,948.82 ÷ 1.015	103,397.85
and again 103,397.85 ÷ 1.015	101,869.81
which is the value required.	

To test this, let us multiply down to maturity:

Value at 2 years	$101,869.81
Income 1½%, ½ year	1,018.70
	509.35
Value at 1½ years, flat	$103,397.86
No coupon.	
Income 1½% ½ year	1,033.98
	516.99
	$104,948.83
Annual coupon paid	4,000.00
Value at 1 year	$100,948.83
Add ½ year's income, 1½%	1,009.48
	504.74
	$102,463.05
Add last ½ year's income, 1½%	1,024.63
	512.32
Total principal and interest	$104,000.00

Annualization

We will now annualize the above process; that is, instead of multiplying twice by 1.015 we will multiply once by 1.030225, which is 1.015 × 1.015, or $(1.015)^2$.

As before, beginning with	$101,869.81
we multiply by 1.030225	3,056.09
	20.37
	2.04
	.51
	$104,948.82
and subtract the coupon	4,000.00
	$100,948.82
again multiply by 1.030225	3,028.47
	20.19
	2.02
	.50
giving the same result	$104,000.00

Thus income has been received on all the investment outstanding at .015 per half year, or .030225 per year.

Semi-Annual Income Annualized

We now, for a purpose, take an ordinary half yearly bond paying .02 twice a year and yielding .015 half yearly, and propose to annualize in this case also. The ratio, annualized, is the same, 1.030225, but there are two coupons of $2,000.00. The first of these, if deferred to the end of the year, must increase to $2,030.00
as it is to earn at .015; the second remains............ 2,000.00

The entire annual cash interest is therefore equivalent to $4,030.00

The processes of multiplying down to maturity the ordinary semi-annual and the annualized will now be given side by side, beginning with the value 101,927.19 found from tables or by calculation. In a third column appears the annualized process for 4% annually.

2% per half year.		4% per year.
Ordinary Process	Annualized Process	Annualized Process
$101,927.19	$101,927.19	$101,869.81
1,019.27	3,057.82	3,056.09
509.64	20.38	20.37
	2.04	2.04
$103,456.10	.51	.51
2,000.00		
$101,456.10		
1,014.56		
507.28		
$102,977.94	$105,007.94	$104,948.82
2,000.00	4,030.00	4,000.00
$100,977.94	$100,977.94	$100,948.82
1,009.78	3,029.34	3,028.47
504.89	20.20	20.19
	2.02	2.02
$102.492.61	.50	.50
2,000.00		
$100,492.61		
1,004.93		
502.46		
$102,000.00	$104,030.00	$104,000.00

Comparison of Annual and Semi-Annual

In each of these columns the proper principal is attained at maturity with its accompanying interest, either actual or annualized. The point to be noted in the second and third columns is that they only differ in the value of the effective cash-rate, one being .0403, the other .04. We may then say that a semi-annual 4% is effectively a 4.03% annual bond, the income in both cases being 3.0225%. In the semi-annual bond, annualized, we have

Article 90

as the interest difference.....$4,030.00 — $3,022.50 = $1,007.50
in the annual bond.......... 4,000.00 — 3,022.50 = 977.50

These interest-differences, 1,007.50 and 977.50, are important because, according to the second rule in Chap. VIII, we have only to multiply the interest differences by the present worth of an annuity of $1 for 2 periods at 3.0225%. We cannot find this present worth from Table IV except by interpolation, but from Table II we can find the present worth of a single dollar for 4 periods at 1.015, which is exactly equivalent to 2 periods at 1.030225.

This value .... .94218423
must, according to Chapter V, be subtracted from...1.00000000

Finding Present Worth

and the remainder.............................. .05781577
must be divided by............................ .030225
The quotient is..................................1.9128460
the present value of an annuity of $1 for 2 periods at 3.0225%.

In the semi-annual bond we must multiply the above present worth by 1,007.50, and in the annual bond by 977.50.
Premium on ordinary bond....1.912846 × 1,007.50 = 1,927.192
Premium on annual bond......1.912846 × 977.50 = 1,869.807
These agree perfectly with the values obtained otherwise 101,927.19
and 101,869.81

Another Example

As another example, take that of a 4% bond to yield 5%, for two years. Evidently this will be at a discount instead of a premium. To annualize the ratio 1.025, multiply it by itself, giving 1.050625, and the annualized rate is................. .050625
from this subtract................................ .04

giving as interest-difference........................ .010625

To find the present worth of an annuity of $1 for 2 (annual) periods at .050625, take from Table II, column 2½, the value for 4 (semi-annual) periods.............................. .90595064
subtract from 1.00000000

.09404936
Divide by .050625; quotient..........................1.8577652
which is the required annuity of $1.
Multiplying this by .010625; 1.8577652|| × .010625 = .01973876
This is the discount, which subtracted from par....1.00000000

gives the value of a $1 bond......................... .98026124

This may be tested by multiplying down to maturity.

× 1.025	.02450653
	1.00476777
× 1.025	.02511919
	1.02988696
	.04
	.98988696
× 1.025	.02474718
	1.01463414
× 1.025	.02536586
	1.04000000
	.04
	1.00

We are now prepared to formulate a rule for valuing an annual bond on a semi-annual basis without reference to the values of a corresponding ordinary bond.

Rule 1
Example:
1.015^2=1.03022

RULE 1. *a.*—Annualize the ratio of increase by multiplying it by itself.

.04—.030225
=.009775

b.—Subtract the rate from the annual coupon, or vice versa, to give the interest-difference.

.009775
×1.9128453
=.0186981

c.—Multiply the latter by the present worth of an annuity of $1 for the number of annual periods at the annualized rate, giving the premium or the discount.

Transforming Semi-Annual Premiums into Annual

Where the values of the ordinary semi-annual bond have already been calculated, as in the bond tables, it will be possible to obtain therefrom the values of the annual bond, with a saving of time. The respective values in the first example are in the proportion of 1,007.50 to 977.50; hence we must divide 977.50 by 1,007.50 to give a multiplier or annualizer, which applied to any premium on the ordinary bond will givc the premium on a corresponding annual bond.

977.50 ÷ 1007.50 = .97022333
1,927.19 × .9702233 = 1,869.81

In the second instance, the tabular value of a 4% bond (semi-annual) for 2 years to yield 5% is.................. .98119013
or a discount of................................. .01880987

Transforming Semi-Annual Discounts into Annual

The annualized ratio of increase is.................1.050625
and the annualized rate is......................... .050625
From this subtract in succession .04 the annual coupon, giving .... .010625
and .0405 the annualized value of two coupons, giving. .010125
Divide the former remainder by the latter .010625 ÷ .010125 = ...1.0493827

This is the Annualizer which is applicable for any number of exact years to transform semi-annual tables into annual.
For example: .01880987 × 1.0493827 =........... .01973875
Subtracted from 1. this gives as the value.......... .98026125
differing from the one already obtained............ .98026124
by one cent on a million dollars owing to decimals rounded off.

Rule 2

RULE 2.—To find the annualizer for any two rates:

a.—Find a first interest-difference as in Rule 1, *a* and *b*.

Example:
.009775

b.—Find a second interest-difference using (instead of the annual coupon) the annualized value of the two semi-annual coupons.

.0403—.030225
=.010075

c.—Divide the first interest-difference by the second, giving the annualizer.

.009775
÷.010075
=.9702233 ||

If in this process the annualized rate is greater than the annual coupon, the result will be a discount instead of a premium, even if it was a premium which was extracted from the table. Thus in a $1,000,000 5% annual bond to net 4.95%; the premium 482.03 × the annualizer .2237313 = 107.18 discount and the value is 999,892.82.

It must be observed that only values for full years can be obtained in either of these ways. An odd half year is a "broken" period, and must be treated as in Chapter VIII.

While this, however, is the manner which would doubtless be followed in buying and selling, a more accurate result would be obtained by using as the half-year value the one obtained by multiplying down at the effectiv rate.

Conventional Process

Thus in a bond at 4 per cent., payable annually, on a 3 per cent. semi-annual basis, the values for two years and one year are:

2 years	$1,018,698.07
1 year	1,009,488.22
Maturity	1,000,000.00

The amortization for the first year is $9,209.85, and for the second $9,488.22. Halving these severally, the values by half-years appear as follows:

	Values.	D_1
2 years	$1,018,698.07	$4,604.92
1½ years	1,014,093.15	4,604.93
1 year	1,009,488.22	4,744.11
½ year	1,004,744.11	4,744.11
Maturity	1,000,000.00	

This result would be in accordance with the conventionally establisht rule that during a period (which is here a year) simple interest shall prevail and the amortization take place proportionately to the number of days.

Scientific Process

But the half-year may, with equal propriety, be considered the period, since the income is on a semi-annual basis. Under this assumption we must multiply down:

Value, 2 years	$1,018,698.07	
× 1,015	10,186.98	
	5,093.49	
Value, 1½ years	$1,033,978.54	flat.
Less accrued interest	20,000.00	
Value, 1½ years	$1,013,978.54	and interest.

Similarly, the value at one-half year is fixed at $1,004,630.54, and the series with differences will appear as follows:

	Values.	D_1
2 years	$1,018,698.07	$4,719.53
1½ years	1,013,978.54	4,490.32
1 year	1,009,488.22	4,857.68
½ year	1,004,630.54	4,630.54
Maturity	1,000,000.00	

In the second half of each year there is less amortization, and consequently more earning than in the former half; but this may be defended on the ground that by the conditions prescribed, interest is compounded semi-annually. The earning power at compound interest must continue to increase until a cash payment; and there is no cash payment at the mid-year.

Values Derived from Tables

This latter form of valuation at mid-years is recommended for comparativ (non-commercial) purposes.

The values at *n*½ years, $1,013,978.54 and $1,004,630.54, may be deduced from the ordinary extended tables by multiplying by the annualizer, with this proviso: that the interest difference must first be temporarily added to the tabular premium or discount before multiplying. Thus, in the case just considered, the excess of .02 over .015 is .005 each half year; or on $1,000,000, $5,000.

To find the value for 1½ years take from the table	$14,561.00
add the interest-difference	5,000.00
giving the multiplicand	19,561.00
which × the annualizer .9702233	= 18,978.54
from which again subtract	5,000.00
giving the value as above	$13,978.54

Successiv Process

In general, when a schedule is to be formed for an annual or a quarterly bond, on a semi-annual basis, it will be found easier after ascertaining the initial value to multiply down to maturity, as that will usually require fewer figures.

Article 97

(101) $25,000 4% bonds, interest payable annually, 8 years to run; what is the price at a 3.70% semi-annual basis?

U of M

(102) What multiplier will annualize the premium on the above bonds as given in the regular bond table?

(103) An offering is made of $30,000 3½% bonds, interest payable annually, of which $10,000 mature in one year and $10,000 each year thereafter. What should be paid for them to produce 3.40% semi-annually?

SOLUTIONS: (101) $25,452.30. (102) .87779704||. (103) $30,040.34.

SECTION 15. BONDS AT TWO SUCCESSIV RATES.

Varying Rates

Occasionally bonds are issued with the agreement that the interest paid shall be at a certain rate for some years and at another rate for the remainder of the time to maturity. For example, a fifty-year bond bearing 4% for 20 years and 5% for the following 30 years. The problem is then to find the price at which they will pay a certain income, say 3.60%.

Each of the two successiv cash rates will cause a premium, and we may calculate these premiums separately.

Calculation of Immediate Premium

The premium caused by the 4% rate will only last 20 years and will then vanish; hence this premium is just the same as that on a plain 4% bond for 20 years to net 3.60%, which we find by calculation or from the Tables to be $56,680.10 on $1,000,000.

Calculation of Deferred Premium

The premium produced by the 5% rate does not take effect immediately, but after 20 years. It is a *deferred annuity*. An annuity for the entire 50 years of the excess interest, 1.40%, or in other words the premium on a fifty year 5% bond to net 3.60%, is.................................. $309,245.37

But during the first 20 years there will be no such premium; we have already charged that at 4%. Hence we must by subtraction eliminate the analogous 5% premium for 20 years, which is..... 198,380.36

leaving a remainder.......................... $110,865.01

which is the premium, or present worth, of the enjoyment of a 5 rate (as against 3.60) commencing 20 years from date and continuing till 50 years from date.

Adding together the two premiums, $56,680.10 and $110,865.01, we have $167,545.11 as the premium which should be paid for the bond.

Method by Values

A simpler way to apply the principle is to add together the

4% value for 20 years	$1,056,680.10
and the 5% value for 50 years	1,309,245.37
	$2,365,925.47
and subtract the 5% value for 20 years	1,198,380.36
giving the value of the composit bond	$1,167,545.11

This procedure has the advantage that it applies alike to bonds at a premium and to those at a discount and automatically allows for that distinction.

Symbols

We may for convenience represent the earlier rate by a and the latter rate by b, i being the net income. We may put m for the number of years at which the rate a prevails, and n for the number of years at b; $m+n$ is the entire time. The rule will then be as follows:

Rule

RULE. To find the value of a bond to yield i per cent when by its terms it pays cash interest at the rate a for m years and thereafter at b for n years, maturing in $m+n$ years. Add together the values of an a bond for m years and that of a b bond for $m+n$ years, and from the sum subtract the value of a b bond for m years.

Illustration

An example at a very short time will illustrate the principle of the rule and will admit of demonstration by multiplying down. A bond for $100,000 paying 5% for 1 year (2 periods) and 6% for 1½ years (3 periods) is to be valued so as to yield 4%.

If the rate were 5% for the entire 2½ years the value would be $102,356.73; if it were 6% it would be $104,713.46. Let us analyse these premiums into their component parts, which are the present worths of excess-interest, $500 and $1000 respectively.

Analysis of Premiums

	5%	6%
½ year	490.196	980.392
1 year	480.584	961.168
Premium one year before maturity	970.780	1941.560
1½ years	471.161	942.322
2 years	461.923	923.846
2½ years	452.866	905.732
Premium 2½ years before maturity	2356.730	4713.460

Any premium is the sum of a certain number of present-worths of 500 or of 1000. But in the double-rate bond, only the first two in the 5% column take effect and the last three in the 6% column, as indicated by the braces placed opposite them.

It is evident that the values producing premiums at the 5% rate amount to $970.78, and that those in the 6% column amount to $2771.90 and that the easiest way to obtain this latter amount is to subtract 1941.56 from 4713.46. Hence the premium is

$$970.78+2771.90=3742.68.$$

The equivalent process by the rule would be

a bond *m* years	100,970.78
b bond *m*+*n* years	104,713.46
	205,684.24
—*b* bond *m* years	101,941.56
	103,742.68

It will be interesting to multiply down to maturity and thus test this result.

		$103,742.68
	+	2,074.85
		$105,817.53
	—	2,500.00
		$103,317.53
	+	2,066.35
		$105,383.88
	—	2,500.00
		$102,883.88
	+	2,057.68
		$104,941.56
	—	3,000.00
		$101,941.56
	+	2,038.83
		$103,980.39
	—	3,000.00
		$100,980.39
	+	2,019.61
		103,000.00
	—	3,000.00
Par		$100,000.00

It will sometimes be the case that in multiplying down, the values will increase for a time and then begin to decrease at the change of rate.

(104) An issue of bonds matures on Jan. 1, 1956. Interest is to be at 5% till Jan. 1, 1926, and thereafter at 6%. What is the price at a 3.60% basis on July 1, 1906?

(105) $10,000 of Waterworks Bonds, 5 years to run; first 3 years at 4%, thereafter at 5%, value to yield 4.40%.

(106) Value of the same bonds to net 4¾%; 5¼%.

SOLUTIONS. (104) $1,413422.66. (105) $9988.49. (106) $9824.98; $9617.04.

SECTION 16. REPAYMENT AND REINVESTMENT.

Repayment

When a loan is payable in equal periodic instalments, each covering the interest and part of the principal, the most obvious way of looking at it is that the principal is gradually paid off; and then we have this aspect:

1. A diminishing principal;
   a diminishing interest-charge, and therefore
   an increasing re-payment.

But precisely the same result may be put in a different point of view by assuming that no payment is made at all until the final date of maturity, at which time the sinking fund or sum of instalments plus interest is just sufficient to pay off the whole debt. This will be the aspect:

Calculated Reinvestment

2. An unchanged principal;
   a uniform interest charge;
   a uniform instalment devoted to reinvestment and allowed to accumulate.

A debt of $1,000, interest at 3% per period, may be extinguished in four periods by uniform instalments of $269.03, as we have already seen.

Aspect No. 1

	Instalment	Interest on Balance	Payment on Principal	Principal Outstanding
				$1,000.00
1.	$269.03	$30.00	$239.03	760.97
2.	269.02	22.83	246.12	514.78
3.	269.03	15.44	253.59	261.19
4.	269.03	7.84	261.19	0.

Here we see the diminishing principal, the diminishing interest-charge and the increasing amortisation.

From the reinvestment point of view, we have:

Aspect No. 2

	Instalment	Interest on Entire Principal	Carried to Sinking Fund	Principal
				$1,000.00
1.	$269.03	$30.00	$239.03	1,000.00
2.	269.02	30.00	239.02	1,000.00
3.	269.03	30.00	239.03	1,000.00
4.	269.03	30.00	239.03	1,000.00

	For Re-investment	Interest on Previous Total	Total Accumulated
1.	$239.03		$ 239.02
2.	239.02	$ 7.17	485.22
3.	239.03	14.55	738.81
4.	239.03	22.16	1,000.00

The working of amortisation of principal in its two aspects as repayment and reinvestment should be carefully studied and the problems in Section 6 should be worked over into schedule form in each aspect.

Integration of Principal

This principle will be found to hold: The "principal outstanding" by the first method + the "total accumulated" by the second method = the original debt.

Aspect No. 1 is based entirely on facts. Without regard to reinvestment, it is certain that the borrower pays and the lender receives the exact rate of interest stipulated for each period on the actual balance due at the beginning of such period, and that balance may be either represented by a single account, or, as suggested in the case of bonds, Chapter XIII, a cost-account and an annulling account.

Use of Aspect No. 2

There are some cases where, especially from the point of view of the debtor, it is desirable to keep in view the entire original sum. One of these cases is where it is impossible or impracticable to diminish or pay off the debt before maturity and accumulation is the only method available. Another is that of a trust where there is an obligation to keep the corpus of the fund intact, and consequently reinvestment in some form is necessary.

But the calculations of reinvestment are prospectiv. They have not the same actuality as those of repayment, but are theoretic estimates of what is expected. Unless a contract has been made to take the instalments off one's hands at a fixed rate, the realization is pretty sure to differ from the anticipation.

Replacement

Article 119

There is a third method which I do not find mentioned in the actuarial treatises, and which I will call replacement to distinguish it from repayment and reinvestment. The successive repayments are transferred to new investments, which are not to accumulate but to furnish new income, helping out the diminished income on the waning principal. I have outlined this procedure under trust funds, but for purposes of comparison will put the materials already used in Aspects 1 and 2 into the replacement form, assuming at first that replacements are so invested as exactly to earn 3%.

Schedule

	1	2	3	4	5	6
	Interest	Payment on Principal	Principal Unpaid	Replacement	Interest on Replacements	Total Income
			$1,000.00			
1.	$30.00	$ 239.03	760.97	$ 239.03		$ 30.00
2.	22.83	246.19	514.78	246.19	$ 7.17	30.00
3.	18.44	253.59	261.19	253.59	14.56	30.00
4.	7.84	261.19	0.	261.19	22.16	30.00
	$76.11	$1,000.00		$1,000.00	$43.89	$120.00

Column 4 of replacements is not accumulativ; its interest is not compounded but is used as income, supplementing that in Column 1. The balance of Column 3 + the total of Column 4 at any point make up $1,000. The two corresponding amounts in Columns 1 and 5 always make up $30 (Column 6). At the close, the original $1,000 has been exactly replaced by the new securities.

Expectations not Always Realized

But, as already remarked, it would seldom happen that exactly 3% would be the rate secured for the replacements, which ought to be of the same grade of security, and availability as the original sum. Let us suppose that the rate of interest was declining so that the first replacement had to be loaned at 2.95, the second at 2.90 and the third at 2.75. Columns 5 and 6 are then the only ones changed:

5	6
Interest on Replacements	Total Income
	$ 30.00
$ 7.05	29.88
14.19	29.63
21.16	29.00
$42.40	$118.51

Here we have the principal intact, and the falling-off is a gradual one affecting the interest. If we had proceeded on plan No. 2, the full predicted interest would have been consumed, but the principal would have been impaired, which is inadmissible. Hence, in cases of this kind, we must use the vanishing principal with actual replacement. The reinvestment scheme is a basis of calculation only and cannot, like the repayment, be reduced to practice.

Proof of Accuracy

It is interesting to note that in the repayment method the work may at any point be tested by a fresh calculation, showing the whole procedure to be coherent and consistent. For instance, in our example the principal at three periods from maturity is $760.97. Treating this as the principal, to find the sinking fund we divide 760.97 by 2.82861, just as for 4 periods we divided 1,000 by 3.7171. This gives the result 269.03 as before for the value of the equivalent annuity, 22.83 as the interest (760.97 × .03), and 246.20 as the first repayment or the constant reinvestment, in either aspect.

Varying Rates of Interest

It must not be supposed that there is at any one moment a single rate of interest prevailing. Considerations of security, convenience and availability give rise to different grades of security and different rates of interest. The prudent investor will probably have at the same time some capital out at high rates and some at low. The money at high rates is not quite so secure, not quite so readily realizable, and requires more effort for the collection of its income. That at low rates is nearer to absolute freedom from risk and from the labor of supervision; it almost automatically collects its own income. The investor will have so planned his investments as to endeavor to preserve a judicious equilibrium between different grades of security, and consequently of income. As his investments are liquidated, he will try to maintain or improve this equilibrium and he will choose his reinvestments from a wide range, some of low revenue but highest safety and others of the contrary qualities.

Replacement Naturally on the Same Grade of Security

It is therefore fallacious to assume that, as an author has said, "on the same day and under the same circumstances money received from any one source may be invested at the same rate as that received from any other source." It may be, but practically it will be invested in the same grade of security as that which it replaces.

Dual Rate for Income and Accumulation

When he assumes great risk or when the supply of loanable capital is temporarily deficient, the lender will exact very high rates or refuse to loan. He has been known to require a high rate and also to demand that the instalments of repayment shall be large enough to secure the higher rate on the entire original loan until fully paid, while in computing reinvestments a lower rate, easily obtained, is used.

Thus, $1,000 is loaned, repayable in 4 instalments, such that the lender will have 5% interest per period on the entire capital, while it will be replaced by accumulating at 3%.

Instalment at Two Rates: Income Rate and Reinvestment Rate

The sinking fund is exactly the same as in our previous example, $239.03. But the instalment is

not 30 + 239.03; 269.03
but 50 + 239.03; 289.03.

The instalment is as much greater as the the interest is greater. The accumulation is precisely the same as heretofore. The instalment provides not only 5% on the money remaining invested, but also 2% (unearned) on that which had been repaid.

An instalment of only $282.01 would pay 5% on the outstanding capital, which would gradually be replaced by 3% investments. Thus it is seen that the borrower has to pay more than 5%; in this instance about 5.6.

Amortisation of Premiums at Dual Rate

This loaning at a dual rate is of so little practical importance, at least in this country, that it would not be worth mentioning, except that a few writers have tried to apply the same principle to the amortisation of premiums. They assume that there is no other way of ascertaining the value of a bond than by laying aside the excess of interest and letting it accumulate till maturity. But this is not at all necessary. The question is, what uniform rate is yielded by each dollar of the investment during the time it is outstanding. When this is ascertained, it can make no difference what is done with the capital after it is returned. As well say that the rate of a series of bonds payable $1,000 each year and issued at par cannot be determined until we know at what rate the amounts were reinvested up to the date of the last maturity. Reinvestment has nothing to do with the yield of the original investment.

Dual Rate Bond Tables

Nevertheless, two authors have constructed tables based upon a dual rate, one a rate of income, the other a rate of accumulation, and they have taken the latter at the arbitrary figure of 4%, irrespective of the grade of the bond.

Modified Method for Valuing Premiums

It is proper to give the method by which these results seem to be obtained, or, at least, a method which will produce those results.

As a preliminary we will again consider the valuation of a premium and in a slightly different way from any yet given.

Article 71

We have seen that a premium on $1 is the present worth of the difference of rates. We may modify this by saying that it is the difference of rates $(c - i) \times$ the present worth of an annuity of $1 (P), which may be found in Table IV. But to multiply by P is the same thing as to divide by $1 \div P$ or $1/P$. Therefore, another expression for the premium is $(c - i) \div (1/P)$. But we found in Chapter VI that a sinking fund $(1/P)$ is the sum of the rent $(1/S)$ and a single interest (i) added together. Therefore we still further modify our expression:

$$\text{Premium} = (c - i) \div (i + 1/S).$$

Rule

RULE. Subtract the income rate from the cash rate and use this as a dividend. Add the instalment from Table V to the rate of income and this will be the divisor. The quotient will be the premium.

EXAMPLE: What is the premium on a 6% bond (semi-annual) for $1, 50 years, to yield 5%?

$c = .03$; $i = .025$; $c - i = .005$ (Dividend).
$1/S$ at 2½% 100 periods $= .002312$ (Table V).
$.025 + .002312 = .027312$ (Divisor).
Premium $= .005 \div .027312 = .18307$.
Value of bond, $1.18307.

To introduce the feature of an accumulativ rate differing from the income rate, it is only necessary to change one term in the above formula. The value of $1/S$ must be taken from the column of Table V, which represents the accumulativ rate, i remaining as the income rate.

EXAMPLE: What is the premium on a 6% bond, as above, to yield 5% on the entire investment to maturity and to replace the principal by a sinking fund at 4%?

Computation at Dual Rate

$c - i = .03 - .025 = .005$ (Dividend).
$1/S$ at 2% $= .003203$ (Table V).
$i + 1/S = .025 + .003203 = .028203$ (Divisor)
$.005 \div .028203 = .17729$.

Value of bond $= 1.17729$, agreeing with Croad's and Robinson's tables.

The constant income is .0294322, which subtracted from the cash received .03 leaves as contribution to the sinking fund .0005678. At 4% an annuity of .0005678 will amount in 50 years to .17729|| as may be ascertained from Table III, thus replacing the premium.

Dual Rate in Bookkeeping

This form of valuation, which introduces an arbitrary element, cannot be satisfactorily applied in the bookkeeping processes of Chapter XIII. It is impossible to derive one value from another consistently. The result will not agree with a fresh calculation. and the profit and loss on a sale will be distorted. Any intermediate value, as shown by the actuaries, may have three different versions.

Utilization of Dual Principle

Article 119

While tables on a fixed replacement rate are useless for purchasing securities, the principle may occasionally be utilized. Thus, the trustee referred to in Chapter VIII may find that it is impracticable to invest favorably such small amounts as $400 or $500 and may conclude to deposit a sinking fund in a savings bank where he may reasonably expect that it will accumulate for the next five years at 3½%, or he may make a contract with a trust company on the same terms. He may then decide also that it is better for the beneficiary to receive a uniform income, rather than one gradually decreasing.

At 3½% accumulation (Table V) a sinking fund of .092375 will amount to $1, therefore by multiplication it will take $414.883 to accumulate $4,491.29 in 10 periods. Out of the coupon of $2,500 must be taken the instalment for $414.88, leaving for the beneficiary $2,085.12 instead of the $2,089.83 with which he would have begun on the replacement plan and which would have gradually fallen to $2,035 for the last half year.

SECTION 17. INSTALLATION OF AMORTISATION ACCOUNTS.

Keeping-Up the System

Introduction of System

When the accounts of securities have once been establisht on the plan of gradual extinction of premiums and discounts, it is not difficult to value each new purchase as it comes and prepare its appropriate schedule, running if desired all the way to the date of redemption. When, however, the accounts have been previously kept on the basis of par or of cost, and it is desired to introduce investment-values instead, the task is much greater.

Calculations Need Not Cover Entire Time

It might be supposed at first thought that it would be necessary to start the schedules back at the date of purchase, but this is entirely unnecessary. For example, we find a 5% bond for $100,000 which 20 years ago was bought for $112,650, and which has 10 years to run. At the date of purchase it must have had 30 years to run. Turning to the table of 5% bonds, 30-year column, we find that this was (within 39 cents) a 4¼% basis. Turning again to the 10-year column, it appears that the value of a 5-year bond at 4¼% is $106,058.46. It is sufficiently accurate to begin with this value disregarding the 39 cents residue, altho that residue might be eliminated by the proportion

12,649.61 : 6,058.46 : : .39 : .19.

This would increase the present value to 106,058.65.

Same Basis to be Retained

So long as the same basis is preserved, any number of intervening years may be disregarded.

The following procedure may be recommended:

List of Securities

1. Make an accurate list of the issues held, giving the following particulars: Date of maturity; dates of purchase; par value of each lot; cost of each lot, being at the rate of ——; rate of interest paid, and a space for inserting the income-basis when ascertained; column for valuation at a date one period earlier than the proposed date of transformation.

Ascertain Basis

2. Ascertain on what income-basis each lot was bought. This is done the easiest by using the tables. In these and the subsequent calculations it will be found advantageous to use blank books and entrust nothing to loose papers. Head each calculation with a statement of the problem which it solves. Paper for these blank books, ruled with down-lines, every third one of which is darker, will much facilitate the work, and it is desirable to have the pages numbered in a continuous series, for reference.

Valuation at Basis

3. Find the value of each lot at the initial date, which is, as already stated, one period earlier than the date on which the books are to be transformed to investment values.

Aggregating Lots

4. Where different lots of the same class have been purchased at various dates and prices, their values at the various bases on the initial date are added together, giving a composit value. Ascertain what is the income-basis for the time yet to run on this composit value. This basis is the average basis for the remaining time of the bond.

Calculations First Period

5. Having carefully verified all the initial values and the effectiv rates, proceed to calculate the amortisation and accumulation of each class for one period, commencing a schedule for each. The resulting values are again verified with care, these being the values with which the new accounts will begin.

Continuing Schedules

6. Continue the calculations of successiv values, carrying them into decimals two places beyond cents, ignoring slight differences in the last figure. Copy the results, rounded to the nearest cent, into the schedules, and complete the latter. If time allows, it is advisable to calculate each schedule to maturity, because no better proof of the correctness of the entire chain of values can be had than the fact that the bond reduces exactly to par at maturity. But if time presses, only a few of the values may be calculated, but the last one should be verified by some independent method. It is well in this case to leave in the blank book sufficient room to complete the calculations for each schedule. A reference on the schedule to the page of the blank book where the calculation is made, will be useful.

Transforming Entries

7. Make such entries as will place the ledger or ledgers on the investment-value basis.

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